

Casimir Effect on the Radius Stabilization of the Noncommutative Torus

Wung-Hong Huang

Department of Physics

National Cheng Kung University

Tainan, 70101, Taiwan

We evaluate the one-loop correction to the spectrum of Kaluza-Klein system for the ϕ^3 model on $R^{1,d} \times (T_\theta^2)^L$, where $1+d$ dimensions are the ordinary flat Minkowski spacetimes and the extra dimensions are the L two-dimensional noncommutative tori with noncommutativity θ . The correction to the Kaluza-Klein mass spectrum is then used to compute the Casimir energy. The results show that when $L > 1$ the Casimir energy due to the noncommutativity could give repulsive force to stabilize the extra noncommutative tori in the cases of $d = 4n - 2$, with n a positive integral.

E-mail: whhwung@mail.ncku.edu.tw

1 Introduction

The Casimir effect is the contribution of a non-trivial geometry on the vacuum fluctuation [1-3]. The corresponding change in the vacuum electric energy is found to attract the two perfectly conducting parallel plates in the original investigation [1]. The property of the attractive Casimir force is also found in the gravitational system [4]. It is proposed to render the extra spaces in the Kaluza-Klein unified theory [5] sufficiently small and thus could not be observed. It is generally believed that the nonperturbative quantum gravity can stabilize the size of the extra spaces, as the minimum length scale, Plank scale, is set in.

Another available scale is the noncommutativity θ_{ij} revealed in the string theory [6-8]. It is proved to arise naturally in the string/M theories [8]. Initially, Connes, Douglas and Schwarz [6] had shown that the supersymmetric gauge theory on noncommutative torus is naturally related to the compactification of Matrix theory [8]. More recently, it is known that the dynamics of a D-brane [9] in the presence of a B-field can, in certain limits, be described by the noncommutative field theories [7].

A distinct characteristic of the noncommutative field theories, found by Minwalla, Raamsdonk and Seiberg [10], is the mixing of ultraviolet (UV) and infrared (IR) divergences reminiscent of the UV/IR connection of the string theory. In a recent paper [11] we found that the noncommutativity does not affect one-loop effective potential of the scalar field theory. However, it can become dominant in the two-loop level and have an inclination to induce the spontaneously symmetry breaking if it is not broken in the tree level, and have an inclination to restore the symmetry breaking if it has been broken in the tree level. In the paper [12] Nam has tried to use the noncommutativity as a minimum scale to protect the collapse of the extra spaces. He use the one-loop Kaluza-Klein spectrum derived by Gomis, Mehen and Wise [13] to compute the one loop Casimir energy of an interacting scalar field in a compact noncommutative space of $R^{1,d} \times T_\theta^2$, where $1 + d$ dimensions are the ordinary flat Minkowski space and the extra two dimensions are noncommutative torus with noncommutativity θ . The correction is found to contribute an attractive force and have a quantum instability. He therefore turns to investigate the case of vector field and find the repulsive force for $d > 5$. We will mention the mistake in there and the above conclusion is thus unreliable.

In this paper we extend the problem to the theory on $R^{1,d} \times (T_\theta^2)^L$ while consider only the scalar field theory, i.e., ϕ^3 theory with extra L two-dimensional noncommutative tori. In section II, we first extend the works of Gomis, Mehen and Wise [13] to evaluate the one-loop correction to the spectrum of Kaluza-Klein system for the ϕ^3 model on $R^{1,d} \times (T_\theta^2)^L$. The correction to the Kaluza-Klein mass formula has the additional term which resembles that of the winding states in the string theory [14], likes as the property found in the $L = 1$ system [13]. In section III, the obtained spectrum is used to compute the Casimir energy. Section IV is used to analyze the effect of the correction to the Casimir energy on the radius stabilization. We show that when $L > 1$, the Casimir energy in the cases of $d = 4n - 2$, with n a positive integral, could give repulsive force to stabilize the extra noncommutative tori. This thus suggests a possible stabilization mechanism for a scenario in Kaluza-Klein theory, where some of the extra dimensions are noncommutative.

2 Kaluza-Klein Spectrum

We consider the noncommutative scalar ϕ^3 theory in $R^{1,d} \times (T_\theta^2)^L$ spacetime described by the following action:

$$S = \int d^{1+d}x \, d^{2L}y \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi \star \phi \star \phi \right). \quad (2.1)$$

The \star operator is the Moyal product generally defined by [10]

$$f(x) \star g(x) = e^{+\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}} f(y)g(z)|_{y,z \rightarrow x}. \quad (2.2)$$

in which $\theta_{\mu\nu}$ is a real, antisymmetric matrix which represents the noncommutativity of the spacetime, i.e., $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. In Eq.(2.1) the coordinates x^0, x^1, \dots, x^d represent the commutative four dimensional Minkowski spacetime. The $2L$ extra dimensions are taken to be the L noncommutative 2-tori T_θ^2 whose noncommutative coordinates are described by

$$[y^1, y^2] = [y^3, y^4] = \dots = [y^{2L-1}, y^{2L}] = i\theta. \quad (2.3)$$

When $L = 1$, this coordinate can be realized in string theory by wrapping a five-brane on a two-torus T^2 with a constant B -field along the torus. The low energy effective four dimensional theory resulting from compactification on a noncommutative space is local and Lorentz invariant, hence it can be relevant phenomenologically [13].

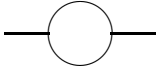
The momentum in the $1+d$ Minkowski spacetime, denoted as p , is a continuous variable. However, the momenta along the tori are quantized as $\vec{k} \rightarrow \vec{k}/R$, where $\vec{k} = (k_1, \dots, k_{2L})$ are the integrals. Therefore, using the Feynman rule [13], which includes the propagator

$$\xrightarrow{p, \vec{n}} \quad \frac{i(1-\delta_{\vec{n},0})}{p^2 - \vec{n}^2 - m^2}$$

and vertex

$$\begin{array}{c} \vec{m} \\ | \\ \swarrow \quad \searrow \\ \vec{k} \quad \vec{n} \end{array} \quad -i\lambda \cos\left(\frac{\theta}{2R^2} \vec{n} \wedge \vec{k}\right) \delta_{\vec{k}+\vec{n}+\vec{m},0}$$

in which $\vec{n} \wedge \vec{k} \equiv (n_1 k_2 - n_2 k_1) + (n_3 k_4 - n_4 k_3) \dots + (n_{2L-1} k_{2L} - n_{2L} k_{2L-1})$, the one loop contributions to the two point functions is



$$\begin{aligned} &= \frac{\lambda^2}{2} \frac{1}{(2\pi R)^2} \sum_{\vec{k}} \int \frac{d^{1+d}l}{(2\pi)^4} \frac{\cos^2(\theta \vec{n} \wedge \vec{k}/(2R^2))(1 - \delta_{\vec{k},0})(1 - \delta_{\vec{n}+\vec{k},0})}{(l^2 - \vec{k}^2/R^2 - m^2)((l+p)^2 - (\vec{n} + \vec{k})^2/R^2 - m^2)} \\ &= \frac{\lambda^2}{4} \frac{1}{(2\pi R)^2} \sum_{\vec{k}} \int \frac{d^{1+d}l}{(2\pi)^4} \frac{1 - 2\delta_{\vec{k},0} - 2\delta_{\vec{n}+\vec{k},0} + \cos(\theta \vec{n} \wedge \vec{k}/R^2)}{(l^2 - \vec{k}^2/R^2 - m^2)((l+p)^2 - (\vec{n} + \vec{k})^2/R^2 - m^2)}. \end{aligned} \quad (2.4)$$

In which l denotes the loop momenta along the noncompact directions, while \vec{k}/R is loop momenta along compact directions. Similarly, $p(\vec{n}/R)$ is the external momenta along the noncompact(compact) directions. The derivation of Eq.(2.4) has used the half angle formula for the cosine and the property that $\vec{n} \wedge \vec{n} = 0$, as detailed by Gomis, Mehen and Wise [13].

The first term, second term and third term in Eq.(2.4) are all divergent [13]. They are irrelevant to our investigation and will not be discussed furthermore. The last term contains a oscillatory factor $\cos(\theta \vec{n} \wedge \vec{k}/R^2)$ which makes the non-planar correction term to be ultraviolet finite [13]. It will be evaluated in below.

We use the Feynman parameter x to perform the integral over the momentum l and then express the result in terms of the Schwinger parameter α . The one-loop self energy evaluated from Eq.(2.4) then becomes

$$\begin{aligned} \Sigma = & -\frac{\lambda^2}{4(4\pi)^{L+\frac{1+d}{2}}} \int_0^1 dx \int_0^\infty d\alpha \alpha^{1-L-\frac{1+d}{2}} \exp \left[-\alpha \left[m^2 + x(1-x)(-p^2 + \frac{\vec{n}^2}{R^2}) - \frac{\theta^2 \vec{n}^2}{4R^2} \right] \right] \times \\ & \left[\frac{1}{2} \Pi_{i=1}^{2L-1} \vartheta(xn_i + i\frac{\theta n_{i+1}}{2\alpha}) \vartheta(xn_{i+1} - i\frac{\theta n_i}{2\alpha}) + \frac{1}{2} \Pi_{i=1}^{2L-1} \vartheta(xn_i - i\frac{\theta n_{i+1}}{2\alpha}) \vartheta(xn_{i+1} + i\frac{\theta n_i}{2\alpha}) \right]. \end{aligned} \quad (2.5)$$

To obtain the above result we have performed the sum over \vec{k} by using the definition of the Jacobi theta function

$$\vartheta(\nu, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2i\pi n \nu), \quad (2.6)$$

and the property of modular transformation

$$\vartheta(\nu, \tau) = (-i\tau)^{-1/2} \exp(-\pi i \nu^2 / \tau) \vartheta(\nu/\tau, -1/\tau). \quad (2.7)$$

When θ is very small the leading contribution will come from the region $\alpha \rightarrow 0$. Thus we can set $\vartheta = 1$ in the Eq.(2.5) and the leading correction to the spectrum of Kaluza-Klein system becomes

$$\Sigma = -\lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2} \right)^{L+\frac{d-3}{2}}, \quad (2.8)$$

in which

$$\lambda_{d,\theta}^2 = \frac{\lambda^2 \Gamma(L + \frac{d-3}{2})}{64 \pi^{L+\frac{d+1}{2}} \theta^{2(L+\frac{d-3}{2})}}. \quad (2.9)$$

The result reduces to that in [13] when $L = 1, d = 3$.

Note that the above result tells us that the correction term will depend on the dimension d . However, the investigation in [12] use the form of $d = 3$ to study the cases in other dimensions and thus draw the wrong conclusion. The same mistake also appears in [12] in investigating the vector field system.

3 Casimir Energy

Casimir energy is obtained by summing up the energy ω of all the modes as follows [1]:

$$\begin{aligned}
u &= \frac{1}{2} \sum_{n_1, \dots, n_{2L}=1}^{\infty} \int \frac{d^d \vec{p}}{(2\pi)^d} \omega_{\vec{n}, \vec{p}} \\
&= \frac{1}{2} \sum_{n_1, \dots, n_{2L}=1}^{\infty} \int \frac{d^d \vec{p}}{(2\pi)^d} \sqrt{\vec{p}^2 + \frac{\vec{n}^2}{R^2} - \lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}}} \\
&= \frac{1}{2} \sum_{n_1, \dots, n_{2L}=1}^{\infty} \int \frac{d^d \vec{p}}{(2\pi)^d} \int_0^\infty \frac{dt}{t} t^{-1/2} e^{-t \left(\vec{p}^2 + \frac{\vec{n}^2}{R^2} - \lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}} \right)} \\
&= \frac{-1}{4\sqrt{\pi}} \frac{1}{(4\pi)^{d/2}} \sum_{n_1, \dots, n_{2L}=1}^{\infty} \int_0^\infty \frac{dt}{t} t^{-(d+1)/2} e^{-t \left(\frac{\vec{n}^2}{R^2} - \lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}} \right)} \\
&= \frac{-1}{4\sqrt{\pi}} \frac{1}{(4\pi)^{d/2}} \Gamma\left(-\frac{d+1}{2}\right) \sum_{n_1, \dots, n_{2L}=1}^{\infty} \left(\frac{\vec{n}^2}{R^2} - \lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}} \right)^{\frac{d+1}{2}}, \tag{3.1}
\end{aligned}$$

in which, for simplicity, we consider the massless case. To obtain the above result we first use the Schwinger's proper time t to handle the square root, then integrate the transverse momentum \vec{p} by doing a Gaussian integral, and finally integrate the Schwinger's proper time by using the integral representation of Gamma function.

In the perturbative regime, $\lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}} \ll 1$, we can approximate Eq.(3.1) by

$$\begin{aligned}
u &= \frac{-1}{4\sqrt{\pi}} \frac{1}{(4\pi)^{d/2}} \Gamma\left(-\frac{d+1}{2}\right) \sum_{n_1, \dots, n_{2L}=1}^{\infty} \left(\frac{\vec{n}^2}{R^2} \right)^{\frac{d+1}{2}} \left[1 - \frac{d+1}{2} \lambda_{d,\theta}^2 \left(\frac{R^2}{\vec{n}^2}\right)^{L+\frac{d-3}{2}} + \dots \right] \\
&= \frac{-1}{4\sqrt{\pi}} \frac{1}{(4\pi)^{d/2}} \Gamma\left(-\frac{d+1}{2}\right) \left[v_{2L}\left(-\frac{d+1}{2}\right) \frac{1}{R^{d+1}} - \lambda_{d,\theta}^2 \frac{d+1}{2} v_{2L}(L-1) R^{2(L-1)} \right], \tag{3.2}
\end{aligned}$$

in which

$$v_N(s) \equiv \sum_{n_1, \dots, n_N=1}^{\infty} \left[\frac{1}{n_1^2 + \dots + n_N^2} \right]^{-s}. \tag{3.3}$$

From Eq.(3.2) and reflection formula [2]

$$\pi^{-s} \Gamma(s) v_N(s) = \pi^{s-\frac{N}{2}} \Gamma\left(\frac{N}{2} - s\right) v_N\left(\frac{N}{2} - s\right), \tag{3.4}$$

we finally find that the Casimir energy can be expressed as

$$u = \frac{-1}{4\sqrt{\pi}} \frac{1}{(4\pi)^{d/2}} \left[\frac{1}{\pi^{L+d+1}} \Gamma\left(L + \frac{d+1}{2}\right) v_{2L}\left(L + \frac{d+1}{2}\right) \frac{1}{R^{d+1}} - \lambda_{d,\theta}^2 \frac{d+1}{2} \Gamma\left(-\frac{d+1}{2}\right) v_{2L}(L-1) R^{2(L-1)} \right]. \quad (3.5)$$

The first term is negative and contribute an attractive force to unstabilize the radius. When the second term, which depends on the noncommutativity θ , is positive then the correction due to the space noncommutativity may stabilize the radius. The compactification radius, if it exists, will be at

$$R = \left[\frac{-\Gamma\left(L + \frac{d+1}{2}\right) v_{2L}\left(L + \frac{d+1}{2}\right)}{\lambda_{d,\theta}^2 \pi^{L+d+1} (L-1) \Gamma\left(-\frac{d+1}{2}\right) v_{2L}} \right]^{\frac{1}{2L+d-1}}. \quad (3.6)$$

In the next section we will use the above results to analyze the stabilization of the noncommutative extra tori.

4 Results and Conclusions

(a) $L = 1$:

Let us first analyze the case of $L = 1$. From Eq.(3.5) we see that when $L = 1$, i.e., extra noncommutative space is a single 2-torus, and d is odd, then the contribution from the correction to the Casimir energy is finite after using the reflection formula Eq.(3.4). The correction term is independent of the radius R of the torus. Thus, up to the order of perturbation there is no stabilization and we have to consider the next order in correction to the Casimir energy. This is contrast to that in [12]. The investigation in [12] use the spectrum of $d = 3$ to study the Casimir effect in other dimensions and thus draws a wrong conclusion that when $d > 3$ the correction will have attractive force to unstabilize the size of radius R .

Note that when $L = 1$ and d is an even number, then the divergent term $v_2(0)$ itself in Eq.(3.5) will lead the Casimir energy to be infinite. This may mean that the approximation

used to derive Eq.(3.2) from Eq.(3.1) is broken down. This seems something strange and the problem is remained to be solved.

(b) $L > 1$:

Because there is no stabilization for a single 2-torus up to the order of perturbation, let us turn to the space with multiple 2-torus. From Eq.(3.5) we see that when $d = 4n$, with n an integral, then $\Gamma(-\frac{d+1}{2}) > 0$ and the correction energy is negative. Thus the correction term will contribute a attractive force and system does not have a stable radius. However, when $d = 4n - 2$, with n a positive integral, then $\Gamma(-\frac{d+1}{2}) < 0$ and the correction energy is positive. Thus the system will have a stable radius. The compactification radius is expressed in the formula Eq.(3.6).

Note that when $L > 1$ and d is an odd number, then the divergent term $\Gamma(-\frac{d+1}{2})$ in Eq.(3.5) will lead the Casimir energy to be infinite. Just likes that in the case of $L = 1$ with odd d , this may mean that the approximation used to derive Eq.(3.2) from Eq.(3.1) is broken down. The problem is remained to be solved.

Finally, let us make some remarks:

(1) In this paper we have seen that when $L > 1$ the Casimir energy of ϕ in the cases of $d = 4n - 2$, with n a positive integral, could give repulsive force to stabilize the extra noncommutative tori. This thus suggests a possible stabilization mechanism for a scenario in Kaluza-Klein theory, where some of the extra dimensions are noncommutative.

(2) The ϕ^3 interaction considered in this paper is like the interaction in the string field theory [15]. Therefore it has some motivations from the string theory despite of the fact that the ϕ^3 theory itself will become divergent if the spacetime is larger then six. However, when the spacetime is larger then six we can assume that some fields, coming from the other modes of the string, will contribute to the loop diagram and make the total system renormalizable. We have in the section III, therefore, calculated the Casimir effect of the ϕ field while neglect others.

(3) The Casimir effect is null in the supersymmetry system, as the contribution of boson field is just canceled by that of the fermion field . However, some mechanisms are proposed to break the supersymmetry to describe the physical phenomena. Thus the remaining Casimir

effect may be used to render the compact space stable. An interesting mechanism to break the supersymmetry is the temperature effect, which is the scenario in the early epoch of the universe. Therefore it is useful to investigate the finite-temperature Casimir effect of ϕ^3 model on $R^{1,d} \times (T_\theta^2)^L$. It will be presented in elsewhere [16].

1. H.B.G. Casimir, “*On the Attraction Between Two Perfectly Conducting Plates*”, Proc. K. Ned. Akad. Wet, **51** (1948) 793; C. Itzykson and J. B. Zuber, *Quantum Field Theory*, New York, McGraw-Hill, 1980.
2. J. Ambjorn and S. Wolfram, Ann. Phys. **147** (1983) 1.
3. G. Pluier, B. Müller, and W. Greiner, Phys. Rep. **134** (1986) 87;
V.M. Mostepanenko and N.N. Trunov, “*The Casimir Effect and its Applications*”, Oxford Univ. Press, 1997.
4. T. Appelquist and A. Chodos, Phys. Rev. Lett. **50** (1983) 141; Phys. Rev. **D28** (1983) 772.
5. Th. Kaluza, Sitzungsber. d. K. Preuss. Akad. d. Wissen. z. Berlin, Phys.-Math. Klasse (1921) 966; O. Klein, Z. F. Physik 37 (1926) 895;
T. Appelquist, A. Chodos, and P.G.O. Freund, “*Modern Kaluza-Klein Theories*”, Addison-Wesley, Menlo Park, 1987.
6. A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative Geometry and Matrix Theory: Compactification on Tori”, hep-th/9711162, JHEP 9802:003 (1998);
B. Morariu and B. Zumino, “Super Yang-Mills on the Noncommutative Torus”, hep-th/9807198;
C. Hofman and E. Verlinde, “U-duality of Born-Infeld on the Noncommutative Two-Torus”, hep-th/9810116, JHEP **9812**, 010 (1998).
7. N. Seiberg and E. Witten, “String Theory and Noncommutative Geometry”, hep-th/9908142, JHEP **9909**, 032 (1999).
8. N. A. Obers and B. Pioline, Phys. Rep. **318** (1999) 113, hep-th/9809039.
9. J. Polchinski, *String Theory*, Cambridge University Press, 1998.
10. S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative Perturbative Dynamics”, hep-th/9912072, JHEP **0002**, 020 (2000);
C. P. Martin and D. Sanchez-Ruiz, “The One-loop UV Divergent Structure of U(1)

- Yang-Mills Theory on Noncommutative R^4 ", hep-th/9903077, Phys. Rev. Lett. **83**, 476 (1999);
- I. Chepelev and R. Roiban, "Renormalization of Quantum field Theories on Noncommutative R^d . I: Scalars", hep-th/9911098;
- B. A. Campbell and K. Kaminsky, "Noncommutative Field Theory and Spontaneous Symmetry Breaking", Nucl. Phys. B 581 (2000) 240, hep-th/0003137;
- A. Matusis, L. Auskind and N. Toumbas, "The IR/UV Connection in Noncommutative Gauge Theories", hep-th/0002075 ;
11. W. H. Huang, "Two-Loop Effective Potential in Noncommutative scalar field theory", hep-th/0009067.
 12. S. Nam, "Casimir Force in Compact Noncommutative Extra Dimensions and Radius Stabilization", hep-th/0008083.
 13. J. Gomis, T. Mehen and M.B. Wise, "Quantum Field Theories with Compact Noncommutative Extra Dimensions", JHEP **0008**, 029 (2000). hep-th/0006160.
 14. W. Fischler, E. Gorbatov, A. Kashani-Poor, S. Paban, P. Pouliot and J. Gomis, "Evidence for Winding States in Noncommutative Quantum Field Theory", hep-th/0002067.
 15. M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, 1986.
 16. W. H. Huang, "Finite-temperature Casimir Effect on the Radius Stabilization of the Noncommutative Torus", in preparation.